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LETTER TO THE EDITOR

A q-analogue of the dual space to the Lie algebras gl(2) and sl(2)

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Abstract. For the Lie group $\mathrm{GL}(n)$, the linear Poisson bracket on the dual space $\mathrm{gl}(n)^*$ to the Lie algebra $\mathrm{gl}(n)$ is quadratically deformed to make the coadjoint action of $\mathrm{GL}(n)$ into a Poisson action, for arbitrary quadratic Poisson Lie structure on $\mathrm{GL}(n)$. For the case $n = 2$, these formulae are quantized for both quantum groups $\mathrm{GL}_q(2)$ and $\mathrm{SL}_q(2)$.

If G is a Lie group with the Lie algebra $\mathcal{G}$ then the coadjoint action of G on the dual space $\mathcal{G}^*$ to $\mathcal{G}$ preserves the natural linear Poisson bracket on $\mathcal{G}^*$. In other words, the action map Ad^*: $\mathrm{G}\times\mathcal{G}^* \to \mathcal{G}^*$ is a Poisson map, with the Poisson bracket on G being zero. If G is a Poisson Lie group then this action, in general, is no longer Poisson unless the Poisson structure on $\mathcal{G}^*$ is suitably modified. For the case $\mathrm{G} = \mathrm{GL}(n)$, this modification takes the form

$$\{N_i^\alpha, N_j^\beta\}_2 = r_{ij}^{\varphi\psi} N_\varphi^\alpha N_\psi^\beta + r_{\varphi\psi}^{\alpha\beta} N_i^\varphi N_j^\psi - r_{i\psi}^{\varphi\beta} N_\varphi^\alpha N_j^\psi - r_{\psi j}^{\alpha\varphi} N_i^\psi N_\varphi^\beta \tag{1}$$

where the quadratic Poisson Lie structure on $\mathrm{GL}(n)$ (and $\mathrm{Mat}(n)$) is

$$\{M_i^\alpha, M_j^\beta\} = r_{ij}^{\varphi\psi} M_\varphi^\alpha M_\psi^\beta - r_{\varphi\psi}^{\alpha\beta} M_i^\varphi M_j^\psi . \tag{2}$$

The (coboundary) form (2) is the most general quadratic multiplicative Poisson bracket on $\mathrm{Mat}(n)$, with an arbitrary skewsymmetric r-matrix $r_{ij}^{\alpha\beta} = -r_{ji}^{\beta\alpha}$; the summation convention over non-fixed repeated indices is in force.

The action

$$N \longmapsto MNM^{-1} \tag{3}$$

preserves the Poisson structure (1). The full Poisson structure on $\mathrm{gl}(n)^*$ is the sum of the quadratic piece (1) and the standard (quasi)linear piece

$$\{N_i^\alpha, N_j^\beta\}_1 = \theta(N_i^\beta \delta_j^\alpha - N_j^\alpha \delta_i^\beta) \tag{4}$$

where θ belongs to the ring of Poisson invariants on $\mathrm{gl}(n)^*$ generated by the elements

$$\{\mathrm{tr}(N^\ell) | \ell \in \mathbb{Z}_+\}. \tag{5}$$

Again, the coadjoint action (3) is a Poisson map for the sum $\{\ ,\ \}_2 + \{\ ,\ \}_1$. This follows from the following observation, which *a posteriori* justifies formula (1).

If a Poisson Lie group G acts on a manifold M then the set of (pre) Poisson structures on M which make this action into a Poisson action is an *affine space.* Indeed, this action is Poisson for the pairs of Poisson brackets

$$(\{\ ,\ \}_1^{\mathrm{G}}\,;\ \{\ ,\ \}_1^{M}) \qquad \text{and} \qquad (\{\ ,\ \}_1^{\mathrm{G}}\,;\ \{\ ,\ \}_2^{M})$$

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$$(\{\ ,\ \}_1^{\mathrm{G}}\,;\ \{\ ,\ \}_1^{M}) \qquad \text{and} \qquad (\{\ ,\ \}_2^{\mathrm{G}}\,;\ \{\ ,\ \}_2^{M})$$

where

$$\{\ ,\ \}_2^{\mathrm{G}} = \{\ ,\ \}_1^{\mathrm{G}} - \{\ ,\ \}_1^{\mathrm{G}} = 0.$$

The Poisson brackets (1) and (4) (both of which can be interpreted symplectically) have the following properties.

(a) These brackets are *compatible*. This means that for the sum

$$\{N_i^\alpha, N_j^\beta\}_3 = \{N_i^\alpha, N_j^\beta\}_2 + \{N_i^\alpha, N_j^\beta\}_1 \tag{6}$$

one has

$$\mathrm{Jac}(3) = \mathrm{Jac}(2) + \mathrm{Jac}(1). \tag{7}$$

Here

$$\mathrm{Jac}(s) = \{\ \{N_i^\alpha, N_j^\beta\}_s, N_k^\gamma\}_s + \text{CP} \tag{8}$$

and 'CP' stands for 'cyclic permutation'. Notice that no assumption is made on the vanishing of Jac(2) and Jac(1).

(b) The elements $\{\mathrm{tr}(N^\ell) | \ell \in \mathbb{N}\}$ are Casimir elements for each of the Poisson brackets (1), (4), (6). Thus, one can pass from $\mathrm{GL}(n)$ to $\mathrm{SL}(n)$ and from $\mathrm{gl}(n)^*$ to $\mathrm{sl}(n)^*$.

(c) Each of the Poisson brackets (1), (4), (6), (2) vanishes at the identity $\{N_i^\alpha = \delta_i^\alpha\}$, $\{M_i^\alpha = \delta_i^\alpha\}$. Hence, the action (3) induces a co-representation of the linearized Lie algebra structures on $\mathrm{gl}(n)^* : \mathrm{gl}(n)_N^* \to \mathrm{gl}(n)_M^* \times \mathrm{gl}(n)_N^*$.

(d) The subspace of diagonal matrices is an Abelian subalgebra in each of the Poisson structures (1), (4), (6).

(e) For the quasiclassical limit of the standard quantum group $\mathrm{GL}_q(n)$, understood as the symmetry of a pair of quantum affine spaces

$$x_i x_j = Q_{ij} x_j x_i \qquad \xi_i \xi_j = -Q_{ji} \xi_j \xi_i \qquad Q_{ij} = q^{\epsilon\,\mathrm{sgn(i-j)}} \tag{9}$$

one has

$$r_{ij}^{k\ell} = \epsilon\,\mathrm{sgn}(\mathrm{i}-\mathrm{j})\,\delta_{\mathrm{ji}}^{\mathrm{k}\ell} \tag{10}$$

$$\begin{aligned}\mathrm{Jac}(2) &= \{\, \{N_i^\alpha, N_j^\beta\}_2, N_k^\gamma\}_2 + \mathrm{CP} \\ &= \epsilon^2(\delta_j^\gamma[N_i^\beta(N^2)_k^\alpha - N_k^\alpha(N^2)_i^\beta] + \delta_j^\alpha[N_i^\gamma(N^2)_k^\beta - N_k^\beta(N^2)_i^\gamma]) + \mathrm{CP}. \qquad (11)\end{aligned}$$

Hence, for $n = 2$ one has Jac(2) = 0. For $n \geqslant 3$, Jac(2) is not identically zero. Thus, for gl$(n)^*$ with $n > 2$ and for the choice (9), (10), the Jacobi identities are not satisfied. The case of gl$(2)^*$ (and sl$(2)^*$) is an exceptional one.

Moreover, this exceptional case can be quantized. Recall the commutation relations for a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_q(2)$$

as follows:

$$\begin{aligned} &ab = q^{-1}ba \qquad ac = q^{-1}ca \qquad bd = q^{-1}db \\ &cd = q^{-1}dc \qquad bc = cb \qquad [a,d] = (q^{-1} - q)bc. \end{aligned}$$

Set

$$N = \begin{pmatrix} x & u \\ y & v \end{pmatrix}.$$

Under the action $N \longrightarrow MNM^{-1}$, the following relations among the matrix elements of N are preserved:

$$\begin{aligned} yu &= uy + (q^{-2} - q^2)h^2 + \mu(q^{-1} + q)h \qquad h = (x - v)/(q + q^{-1}) \\ yx &= \left(\frac{2q^2}{1+q^2}x + \frac{1-q^2}{1+q^2}v\right) y - \mu y \\ yv &= \left(q^2\frac{1-q^2}{1+q^2} + \frac{1+q^4}{1+q^2}v\right)y + q^2\mu y \\ ux &= \left(\frac{1+q^{-4}}{1+q^{-2}}x + q^{-2}\frac{1-q^{-2}}{1+q^{-2}}v\right)u + q^{-2}\mu u \\ uv &= \left(\frac{1-q^{-2}}{1+q^{-2}}x + \frac{2q^{-2}}{1+q^{-2}}v\right)u - \mu u \\ xv &= vx. \end{aligned} \qquad (12)$$

Here μ is an arbitrary constant or an element of the ring of invariants (see (c) below). Formulae (12) have the following properties:

(a) The ring $C < y, u, x, v >$ has the PBW property. This is readily seen by applying the diamond lemma to the order $y > u > x > v$.

(b) The q-trace

$$\mathrm{Tr}_q(N) = qx + q^{-1}v \qquad (13)$$

is a central element and an invariant of the conjugation by $M \in \mathrm{GL}_q(2)$. Decomposing N as

$$\begin{pmatrix} x & u \\ y & v \end{pmatrix} = N = \overline{N} + \frac{1}{q+q^{-1}} \mathrm{Tr}_q(N) \qquad \overline{N} = \begin{pmatrix} q^{-1}h & u \\ y & -qh \end{pmatrix} \tag{14}$$

we find the commutation relations between the matrix elements of $\overline{N}$:

$$\begin{aligned} [y, u] &= (q^{-2} - q^2)h^2 + \mu(q^{-1} + q)h \\ yh &= q^2 hy - q\mu y \\ uh &= q^{-2} hu + q^{-1}\mu u \end{aligned} \tag{15}$$

describing $\mathrm{sl}_q(2)^*$.

(c) The elements $\{\mathrm{Tr}_q(N^\ell) | \ell \in \mathbb{N}\}$ are central and invariant. This follows from the following q-analogue of the Cayley–Hamilton theorem for the matrix $\overline{N}$ (14):

$$\overline{N}^2 + \mu\overline{N} = C\mathbb{1} \tag{16}$$

where the quadratic Casimir C is given by the formula

$$C = yu + q^2h^2 - \mu qh = uy + q^{-2}h^2 + \mu q^{-1}h. \tag{17}$$

The element C is central since

$$[C, \overline{N}] = \left[\overline{N}^2 + \mu\overline{N}, \overline{N}\right] = 0. \tag{18}$$

(d) The relations (15) are invariant with respect to the involution

$$u \to y \qquad y \to u \qquad h \to -h \qquad q \to q^{-1}. \tag{19}$$

Remark. The real forms of formulae (12), (15) can be interpreted as a q-analogue of quaternions and the double cover homomorphism SU(2) → SO(3). This will be elaborated upon in another spacetime.